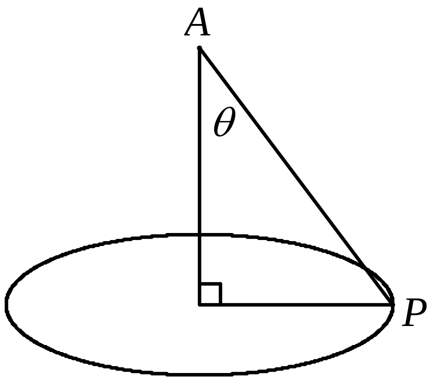
Year 12 Term 2 Mathematics Extension 2, 2009

Question 1

(a) Graph the rectangular hyperbola $xy = 25$ clearly showing the equations of the asymptotes and directrices, and the location of the foci.

(b) From a fixed point *A*, a particle *P* of mass 10 grams is suspended by a light inextensible string of length 17 cm as shown.



The particle *P* rotates with angular velocity ω radians per second in uniform circular motion radius 8cm in the horizontal plane, and has a semi-vertical angle θ with point A .

By resolving the forces on the particle *P* find the angular velocity ω (to 2 significant figures) when the acceleration due to gravity is is $10\,m/s^2$.

(c) A point which is initially at rest at the origin, moves to the right with velocity *v m/s*.
The displacement *x metres* is given by **:**

$$x = v - \tan^{-1} v$$

(i) Find $\ddot{x}$ in terms of *v*.

(ii) Find an expression for the velocity *v* in terms of time *t*.

(d) The region bounded by $y=0$, $x = \dfrac{\pi}{4}$ and the curve $y = \tan^{-1} x$ is rotated about the *y* axis.

(i) Using the method of cylindrical shells show that the volume of revolution *V* is given by **:**

$$V = 2\pi \int_0^{\frac{\pi}{4}} x \tan^{-1} x \, dx$$

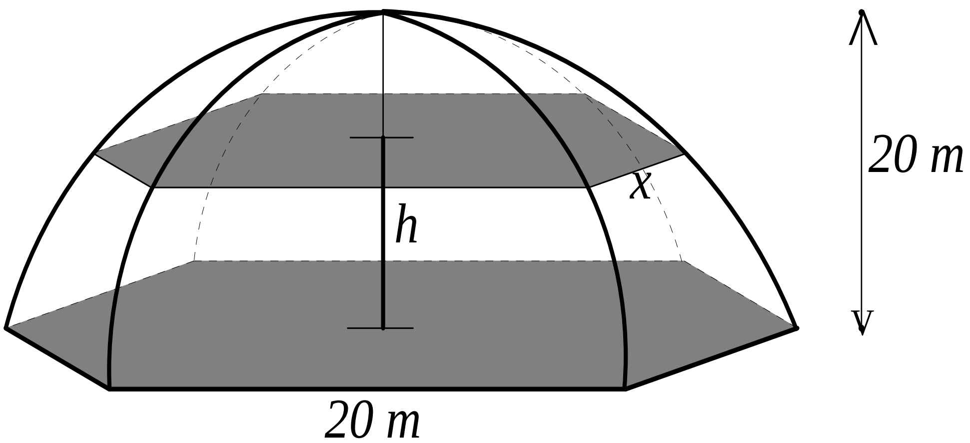
(ii) Hence find the volume of revolution V .

Question 2
From the ground a body of unit mass is projected vertically through a resistive medium.
The resistive force in Newtons is $\frac{1}{20}$ of the speed of the body.
The acceleration due to gravity is $10\,m/s^2$ and the initial velocity is $120\,m/s$.

(a) For the upward motion **:**
  (i) Draw a diagram showing all the forces acting on the body.
  (ii) Show that **:**
  $$\ddot{x} = -10 - \frac{v}{20}$$ where v is the velocity of the body.
  (iii) Show that the maximum height reached is $\approx$ 520 metres .
  (iv) Show that the time T seconds to reach the maximum height is ≈ 9.4 seconds.

(b) The body now falls for T seconds.
  (i) Draw a diagram showing all the forces acting on the body.
  (ii) Find the terminal velocity.
  (iii) Find the velocity v (to 2 significant figures) of the body at T seconds.
  (iv) Find the distance (to 3 significant figures) the body is above the ground after T seconds.

Question 3
(a) Asymmetrical dome shown has a height of 20 metres and a regular hexagonal base of side 20 metres. The apex of the dome is directly above the centre of the base.
Each strut is a quarter of a circle starting from the corner of the base to of the apex of the dome.



(i) For the slice h metres above and parallel to the base, show that the length x of each side of the slice is given by **:**
$$x = \sqrt{400 - h^2} \text{ metres.}$$
(ii) Show that the area A of the slice described above is given by **:**
$$A = \frac{3\sqrt{3}}{2}\left(400 - h^2\right)$$
(iii) Hence calculate the volume of the dome.

(b) The point $T\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Derive the equation of the normal at *T*.

(ii) The normal at *T* meets the rectangular hyperbola $x^2 - y^2 = a^2$ at *Q* and *R*.

For $0 < t < \frac{a}{c}$ show that *T* is the midpoint of QR.

(iii) The rectangular hyperbola $xy = c^2$ is rotated $\frac{\pi}{4}$ radians anti-clockwise about the origin.

(α) Derive the new Cartesian equation for the hyperbola.

(β) State the new equations for the asymptotes and directrices.

Question 4

(a) The region bounded by the *x* axis, the line $x = \frac{\pi}{4}$ and the curve $y = \tan x$ is rotated about the line $y = -1$.

Find the volume *V* of revolution.

(b) On a smooth horizontal table a body with a 2 kg mass is attached to a spring which is pulled to a position *x*.

The force *F* Newtons exerted by the spring towards the origin is given by**:** $F = 26x + 6x^2$ where *x* metres is the displacement of the mass measured from a fixed point O at time *t* seconds.

(i) Draw a force diagram showing all the forces on the body.

(ii) If the body has a velocity of v m/s and is initially at rest 1 metre to the right of O, show that **:** $v^2 = 15 - 13x^2 - 2x^3$.

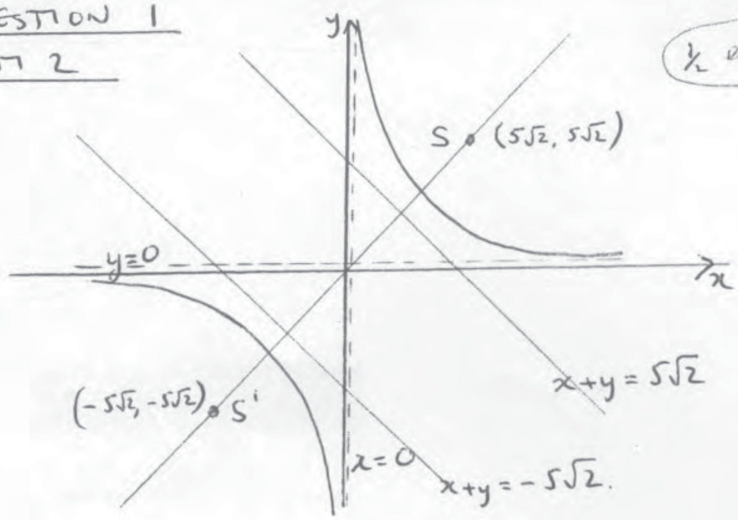
(iii) Find the displacement when the body next comes to rest.

(iv) Find the maximum speed of the body during the motion.

(v) Describe the motion of the body ,showing that the body oscillates between two points.

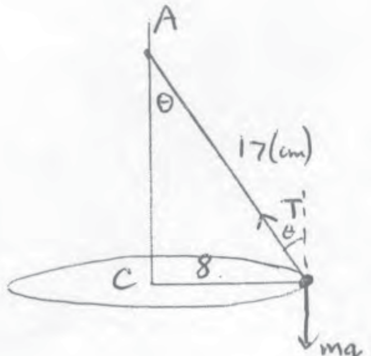
EXT 2 QUESTION 1
2009 TERM 2

1 a) $xy = c^2$
$c = 5$



½ each

b)



$$\left.\begin{aligned} T\cos\theta &= mg \\ T\sin\theta &= m\omega^2 r \end{aligned}\right\} \text{Dividing } \tan\theta = \frac{\omega^2 r}{g} = \frac{8}{15}$$

$\therefore \omega = \sqrt{\dfrac{8g}{15r}}$ rad/sec. $r = 0.08$ (m)
$g = 10$

$\therefore \omega = \sqrt{\dfrac{80}{1.2}} = \underline{\underline{8.2 \text{ rads/sec}}}$ (2 S.F)

④ −1 off each error

c) i) $x = v - \tan^{-1} v$

$\dfrac{dx}{dv} = 1 - \dfrac{1}{1+v^2} = \dfrac{v^2}{1+v^2}$ ①

$\therefore \dfrac{dv}{dx} = \dfrac{1+v^2}{v^2}$

$\ddot{x} = v\dfrac{dv}{dx} = \dfrac{1+v^2}{v} = \underline{\underline{\dfrac{1}{v} + v}}$ ①

ii) $\dfrac{dv}{dt} = \dfrac{1}{v} + v = \dfrac{1+v^2}{v}$

$\therefore \dfrac{dt}{dv} = \dfrac{v}{1+v^2}$

$t = \frac{1}{2}\ln(1+v^2) + c$ ①

When $t = 0$, $v = 0$ $\therefore c = 0$

$\therefore t = \frac{1}{2}\ln(1+v^2)$

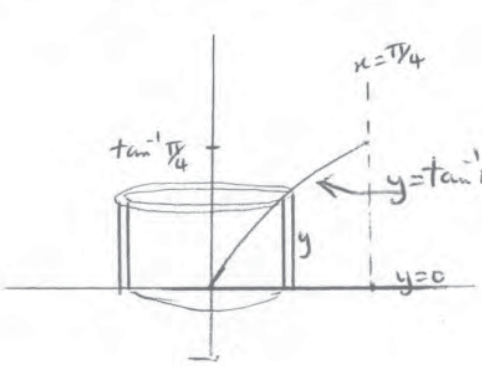
$\therefore 1+v^2 = e^{2t}$

$v^2 = e^{2t} - 1$

$\underline{v = \sqrt{e^{2t}-1}}$ (

(moves to right)

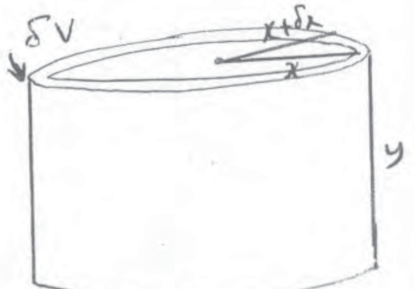
d)



$\delta V \doteqdot \left(\pi(x+\delta x)^2 - \pi x^2\right) y$

$\delta V \doteqdot 2\pi xy\, \delta x$ ①

(neglecting second order terms).



$\therefore V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{\pi/4} 2\pi xy\, \delta x$

$= 2\pi \displaystyle\int_0^{\pi/4} xy\, dx = \underline{\underline{2\pi \int_0^{\pi/4} x\tan^{-1}x\, dx}}$ ①

$V = 2\pi \left[\dfrac{x^2\tan^{-1}x}{2}\right]_0^{\pi/4} - 2\pi \displaystyle\int_0^{\pi/4} \dfrac{x^2}{2(1+x^2)}\, dx$

$= 2\pi \left[\dfrac{x^2\tan^{-1}x}{2}\right]_0^{\pi/4} - \pi\left[x - \tan^{-1}x\right]_0^{\pi/4}$ ①

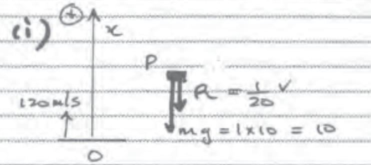
$= 0.914\ u^3$ (3dp)

$= 2\pi \cdot \dfrac{\pi^2}{32}\tan^{-1}\dfrac{\pi}{4} - \dfrac{\pi^2}{4} + \pi\tan^{-1}\dfrac{\pi}{4} = \underline{\underline{\dfrac{\pi(\pi^2+16)\tan^{-1}\frac{\pi}{4}}{16} - \dfrac{\pi^2}{4}}}$ unit^3 ①

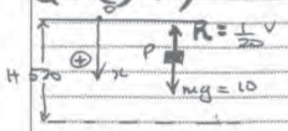
$= 0.914\ u^3$ (3 dp)

MATH EXT 2 SOLUTIONS
TERM 2, 2009

MATHEMATICS Extension 2: Question 2

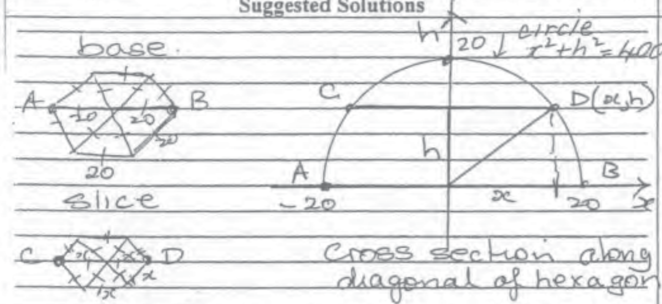
Suggested Solutions	Marks	Marker's Comments
Q 2(a) (i) mass = 1 kg  $\boxed{1}$		½ For $^{+}\uparrow$ and $\downarrow_{mg}\downarrow_{R}$ ½ For $R = \frac{1}{20}v$
(ii) Force equation $F = m\ddot{x} = -mg - R$, $m=1$ (Newtons 2^{nd}) i.e. $1.\ddot{x} = -1\times 10 - \frac{1}{20}v$ $\therefore \ddot{x} = -10 - \frac{1}{20}v$ $\boxed{1}$		1 For evidence $-\frac{1}{2}$ if $-\frac{mV}{20}$ and $\div m$
iii) $\ddot{x} = v\frac{dv}{dx} = -\frac{1}{20}(200+v)$ when $t=T$ $x=H$ and $v=0$ $t=0$ $x=0$ $v=120$ $\int_{120}^{0}\frac{v\,dv}{200+v} = -\frac{1}{20}\int_0^H dx$ $\int_{120}^{0}\left(1-\frac{200}{200+v}\right)dv = -\frac{x}{20}\Big]_0^H = -\frac{H}{20}$ ✓ $\left[v - 200\ln\lvert 200+v\rvert\right]_{120}^{0} = -\frac{H}{20}$ as $v>0$ $200+v>0$ $0\le v\le 120$ $\therefore \left[v - 200\ln(200+v)\right]_{120}^{0} = -\frac{H}{20}$ ✓ $[(0-200\ln 200) - (120 - 200\ln 320)] = -\frac{H}{20}$ $\therefore H = 20\left[120 + 200\ln\frac{200}{320}\right] = 20\left[120 + 200\ln\frac{5}{8}\right]$ $= 519.985483\ldots$ ✓ $\therefore$ max height $= 520$ m $\boxed{3}$	 1 1 ½ ½	(½) $c = 20[120 - 200\ln 320]$ $v = 20\left[120 - v + 200\ln\left(\frac{200+v}{320}\right)\right]$ (2) show evidence
(iv) $\ddot{x} = \frac{dv}{dt} = -\frac{1}{20}(200+v)$ $\boxed{2}$ From 3U: $v = -200 + Ae^{-\frac{t}{20}}$ $t=0$ $v=120$ gives $A = 320$ $v = -200 + 320e^{-\frac{t}{20}}$ so $t=T$ $v=0$ $0 = -200 + 320e^{-\frac{T}{20}}$ ✓ will give $T = 20\ln\frac{320}{200} = 200\ln\frac{8}{5} = 9.400072583\ldots$ $\therefore$ Time $\doteqdot 9.4$ sec **OR** $\int_{120}^{0}\frac{dv}{200+v} = -\frac{1}{20}\int_0^T dt$ $\ln(200+v)\Big]_{120}^{0} = -\frac{T}{20}$ ✓ $T = -20\ln\frac{200}{320} = 20\ln\frac{320}{200} = $ etc	 1 1	from "Further G+D" DE

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
Q 2(b) (i)  $\boxed{1}$	1	1 For direction $\downarrow\oplus$ and $\uparrow R$
(ii) $m\ddot{x} = mg - R$ (Newtons 2^{nd}) i.e. $1\ddot{x} = 10 - \frac{1}{20}v$ $m=1$ i.e. $\ddot{x} = 10 - \frac{1}{20}v$ $\boxed{1}$ For terminal velocity $\lim_{t\to\infty}\ddot{x} = 0$ so $10 - \frac{V_T}{20} = 0$ $V_T = 200$ $\therefore$ terminal velocity is 200 m/s ✓		early $v = 200(1-e^{-\frac{t}{20}})$ $t\to\infty$ $v\to 200^-$
(iii) $\ddot{x} = \frac{dv}{dt} = \frac{1}{20}(200-v)$ and $0\le v<200$ * From 3U: $v = 200 + Be^{-t/20}$ $t=0$ $v=0$ gives $B = -200$ $\therefore v = 200(1-e^{-t/20})$ ✓ so when $t = T \doteqdot 9.4$ $v = 200(1 - e^{-0.47}) = 74.99954634\ldots$ $v = 75$ (2SF) $\therefore$ velocity is 75 m/s **OR** $\int_0^V\frac{dv}{200-v} = \frac{1}{20}\int_0^T dt$ when $t=0$ $v=0$, $t=T$ $v=V$ $-\ln\lvert 200-v\rvert\Big]_0^V = \frac{T}{20}$ * but $200-v>0$ as $0\le v<200$ $\lvert 200-v\rvert = 200-v$ $-\left[\ln\frac{(200-v)}{200}\right] = \frac{T}{20} = \frac{9.4\ldots}{20} = 0.47\ldots$ $\frac{200-v}{200} = e^{-0.47}$ $v = 200(1-e^{-0.47}) = $ etc	 1 1	$T = 200\ln\frac{8}{5}$ 1 For $v = 200(1-e^{-\frac{t}{20}})$ 1 For velocity is 75 m/s (2SF) * big problems if have $v-200$!!! $\boxed{2}$
(iv) From (iii) $v = \frac{dx}{dt} = 200(1-e^{-\frac{t}{20}})$ $\int_0^x dx = 200\int_0^T(1-e^{-\frac{t}{20}})\,dt$ $x\Big]_0^x = x = 200\left[t + 20e^{-\frac{t}{20}}\right]_0^T$ $= 200\left[(T + 20e^{-\frac{T}{20}}) - (0+20)\right]$ $= 200[9.4 + 12.5000\ldots - 20] =$ $= 380.0090731\ldots$ $x = 380$ (3SF) ✓ **OR** $\frac{v\,dv}{dx} = \frac{1}{20}(200-v)$ $t=0$ $x=0$ $v=0$; $t=T$ $x=x$ $v=75$ $\int_0^{75}\frac{v\,dv}{200-v} = \frac{1}{20}\int_0^x dx$ $\int_0^{75}\left(-1 + \frac{200}{200-v}\right)dv = \frac{x}{20}$ ✓ $\left[-v - 200\ln(200-v)\right]_0^{75} = \frac{x}{20}$ ✓ will give $x = -20\left[75 + 200\ln\frac{5}{8}\right]$	 1 1 1	$t=0$ $x=0$ $t=T$ $x=x$ $T = 200\ln 1.6 \doteqdot 9.4$ $-v+200\,\big)\,{-1}\ v$ $v - 200$ $0 + 200$ $\boxed{4}$ $\therefore$ distance above the ground $= H - 380$ $=$ $= 140$ m (3SF) H x ← distance.

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments



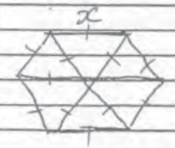
By Pythagras or equation of circle

$$x^2+h^2=400$$

$$x=\sqrt{400-h^2}$$

(2) Must show how hexagons and semi circles are related.

(ii)



6 equilateral triangles of side x

internal angle $=60°$

Area is $6\times\frac{1}{2}x^2\times\sin 60$

$$A=3\left(\sqrt{400-h^2}\right)^2\times\frac{\sqrt3}{2}$$

$$A=\frac{3\sqrt3}{2}\left(400-h^2\right)$$

(2) Must show a triangle area formula.

(iii) HENCE

$$\delta V \doteqdot A\,\delta h$$

$$V \doteqdot \lim_{\delta h\to 0}\sum_{h=0}^{20}\frac{3\sqrt3}{2}\left(400-h^2\right)\delta h$$

$$=\int_0^{20}\frac{3\sqrt3}{2}\left(400-h^2\right)dh$$

$$=\frac{3\sqrt3}{2}\left[400h-\frac{h^3}{3}\right]_0^{20}$$

$$=8000\sqrt3$$

Volume is $8000\sqrt3$ m^3

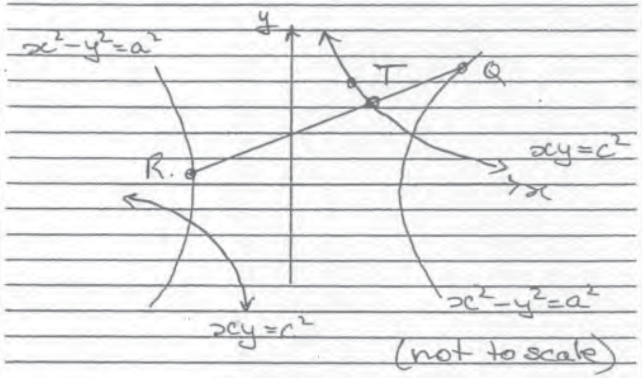
(3) As it is 'HENCE', (1) must show use of δV and $\lim_{h\to0}\sum$

(1) Integral

(1) correct answer.

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments



(i) $xy=c^2$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=\frac{-y}{x}$$

at $T\left(ct,\frac{c}{t}\right)$ $\quad m_T=\frac{-c/t}{ct}=-\frac{1}{t^2}$

$$m_N=t^2$$

equation of Normal

$$y-\frac{c}{t}=t^2(x-ct)$$

$$y=t^2x+c\left(\frac{1}{t}-t^3\right)$$

(1) gradient of tangent

(1) equation of normal (any form)

(ii) intersection with $x^2-y^2=a^2$

$$x^2-\left(t^2x+c\left(\tfrac{1}{t}-t^3\right)\right)^2=a^2$$

$$(1-t^4)x^2-2ct(1-t^4)x-c^2\left(\tfrac{1}{t}-t^3\right)^2-a^2=0$$

midpoint of Q & R $=\frac{-b}{2a}$

$$=\frac{2ct(1-t^4)}{2(1-t^4)}$$

$$=ct$$

as y is on normal

$$y=t^2\times ct+\frac{c}{t}-ct^3$$

$$=c/t$$

$\therefore$ midpoint

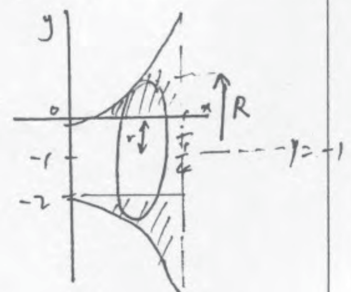
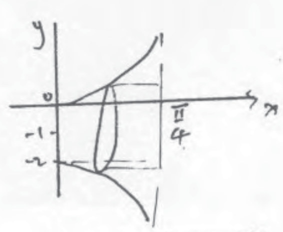
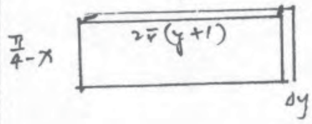
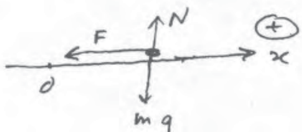
(1) equation in t for point of intersection

($\frac12$) x value

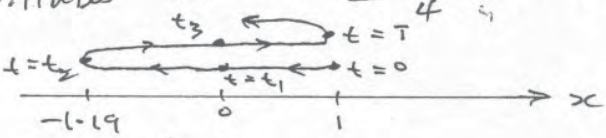
($\frac12$) y value.

Y12 ME2 T2, 2009

T2 2009 4L **MATHEMATICS: Question** 4

Suggested Solutions	Marks	Marker's Comments
a) Disc Method		
$\Delta V \doteqdot \pi[(1+\tan x)^2 - 1^2]\Delta x = \pi(2\tan x + \tan^2 x)\Delta x$	1	
$V = \pi\int_0^{\frac{\pi}{4}} 2\tan x + \tan^2 x \, dx$	1	
$= \pi\left[-2\ln(\cos x) + \tan x - x\right]_0^{\frac{\pi}{4}}$	1	
$\underline{\text{Vol}} = \pi\left[1 - \frac{\pi}{4} + \ln 2\right]$ unit^3	1	
Shell Method		
$\Delta V \doteqdot 2\pi(y+1)(\frac{\pi}{4} - x)\Delta y$	1	
$V = 2\pi\int_0^1 (y+1)[\frac{\pi}{4} - \tan^{-1} y)\,dy$	1	
$V = \frac{3\pi^2}{4} - (\pi^2 - \pi - \pi\ln 2)$	1	
Vol $= \pi\left[1 - \frac{\pi}{4} + \ln 2\right]$ unit^3	1	
b i) 		$\frac{1}{2}$m N $\frac{1}{2}$m mg 1m F towards origin (no forces no marks)
ii) $-m\ddot{x} = 26x + 6x^2 \quad (m=2)$		
i.e. $v\frac{dv}{dx} = -13x - 3x^2$	1	
$\int_0^v v\,dv = \int_1^x -13x - 3x^2\,dx$		
$v^2 = 15 - 13x^2 - 2x^3$	1	Many fudging answers
iii) $v^2 = (1-x)(2x^2 + 15x + 15)$	1	
$x \neq 1 \quad x = \frac{-15 \pm \sqrt{105}}{4} \quad (-1.19, -6.31)$	$\frac{1}{2}$	
next stop at $x = \frac{-15 + \sqrt{105}}{4}$	$\frac{1}{2}$	

~~MATHEMATICS~~: Question 4

Suggested Solutions	Marks	Marker's Comments
biv) For max V $\quad v\frac{dv}{dx} = 0$ $-13x - 3x^2 = 0$ $-x(13 + 3x) = 0$ $\therefore x = 0$ or $\frac{-13}{3}$	$\frac{1}{2} + \frac{1}{2}$	
when $x = -\frac{13}{3}$ $\quad v^2 = -66\frac{10}{27}$ $\quad \therefore x \neq -\frac{13}{3}$	$\frac{1}{2}$	many students forgot to justify $x \neq \frac{-13}{3}$
$x = 0$ $\quad v = \pm\sqrt{15}$ $\therefore$ max speed is $\sqrt{15}$ m/s	$\frac{1}{2}$	
v) At $x=1$, $v=0$ $\quad \ddot{x} = -16$ m/s² $x=0$, $v=\sqrt{15}$ $\quad \ddot{x} = 0$ $x = \frac{-15+\sqrt{105}}{4}$ $\quad \ddot{x} = 11.2$ (m/s²)		
— Initially starts at rest from $x = 1$ m moving left	($\ddot{x} = -16$)	$\frac{1}{2}$m
— increasing speed reaching max of $\sqrt{15}$ m/s at origin, slow down & stops at $x = \frac{-15+\sqrt{105}}{4}$	$\frac{1}{2}$m	
— Turn around moving right ($\ddot{x} = 11.2 > 0$) stops at $x = 1$	$\frac{1}{2}$m	
— oscillates between $\frac{-15+\sqrt{105}}{4} \doteqdot -1.19$ and $x = 1$ 	$\frac{1}{2}$m	Students got the end pts wrong (from previous parts) max $1\frac{1}{2}$m only
vi) $T = 2\int_{\frac{-15+\sqrt{105}}{4}}^{1} \frac{dx}{\sqrt{15 - 13x^2 - 2x^3}}$ $\frac{dx}{dt} = \sqrt{15 - 13x^2 - 2x^3}$ for $t_2 \le t \le T$	1 m	$-\frac{1}{2}$m for each mistake badly done